



Twisted mass chiral perturbation theory for $2 + 1 + 1$ quark flavours

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Abstract

We present results for the masses of pseudoscalar mesons in twisted mass lattice QCD with a degenerate doublet of u and d quarks and a non-degenerate doublet of s and c quarks in the framework of next-to-leading order chiral perturbation theory, including lattice effects up to $\mathcal{O}(a^2)$. The masses depend on the two twist angles for the light and heavy sectors. For maximal twist in both sectors, $\mathcal{O}(a)$ -improvement is explicitly exhibited. The mixing of flavour-neutral mesons is also discussed, and results in the literature for the case of degenerate s and c quarks are corrected.

KEYWORDS: Chiral Lagrangians, QCD, Lattice QCD, Lattice Quantum Field Theory

1 Introduction

For non-perturbative studies of Quantum Chromodynamics on a space-time lattice the formulation with a chirally rotated mass term [1, 2] has proven to be a very effective framework. This so-called twisted mass lattice QCD implies automatic $\mathcal{O}(a)$ Symanzik-improvement if the twist angle is set to a value of $\pi/2$ [3, 4]. Numerical simulations

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of twisted mass lattice QCD have allowed to determine a number of physically relevant hadronic parameters, notably the Gasser-Leutwyler low-energy constants, to a high precision.

First studies of twisted mass lattice QCD have been made with two mass-degenerate quark flavours, representing the up and down quarks, and have allowed to obtain results for quantities in the pion sector of QCD [5, 6, 7], for a review see [8]. In recent work these calculations have been extended to $2 + 1 + 1$ quark flavours, meaning a degenerate doublet of up and down quarks and a non-degenerate doublet of charm and strange quarks [9, 10, 11, 12]. The chiral twist is implemented with two independent twist angles in the up-down sector and in the strange-charm sector. Both of these angles are tuned to $\pi/2$. In the numerical simulations the mass of the charm quark is on the cut-off scale, whereas the mass of the strange quark is near its physical value, so that physical results about observables in the sector of the three lightest quarks can be obtained.

For the analysis of the numerical data, which are obtained at varying quark masses, chiral perturbation theory is an invaluable tool, for a review see [13]. Chiral perturbation yields analytical formulae for the dependence of physical quantities like meson masses and decay constants on the quark masses. It amounts to an expansion around the chirally symmetric limit of vanishing quark masses, and is reliable for sufficiently small quark masses.

For the case of twisted mass lattice QCD, chiral perturbation theory has first been developed and used for $N_f = 2$ quark flavours [14, 15, 16, 17]. It has been extended to $N_f = 3$, including the strange quark, in [18]. In view of the numerical work on $N_f = 2 + 1 + 1$ twisted mass lattice QCD, it is the purpose of this article to present chiral perturbation theory for this situation.

Of course, at this point the question of the region of applicability of chiral perturbation arises. Chiral perturbation theory is well applicable to quark masses as light as those of the up and down quarks. The mass of the strange quark has turned out to be at the border of the region of validity of chiral perturbation theory, depending on the order of the expansion and the use of partial resummations. The charm quark is definitely too heavy to be treated within chiral perturbation theory.

Therefore, chiral perturbation for twisted mass lattice QCD with $N_f = 2 + 1 + 1$ quark flavours is not applicable to quantities in all 4 quark sectors. In this article we restrict ourselves to quantities in the pion- and the kaon-sector. Moreover, for physical values of the strange quark mass, the formulae cannot be applied to obtain precise quantitative results in the kaon sector. Instead, the meaning of these calculations is different. Primarily, the idea is to explicitly reveal the structure of $N_f = 2 + 1 + 1$ twisted mass lattice QCD in the sense of its dependence on the two independent twist angles. Chiral perturbation displays how the values of the two twist angles affect the observables in the different mesonic sectors, and explicitly exhibits the requirements for automatic $\mathcal{O}(a)$ improvement. Secondly, chiral perturbation theory yields the formulae that would allow to

extract $N_f = 3$ low-energy constants from data in the region of unphysically small strange quark masses.

In this letter we present the results of chiral perturbation theory for $N_f = 2 + 1 + 1$ twisted mass lattice QCD in next-to-leading order. Lattice effects are included up to order a^2 . The special case $N_f = 2 + 2$ with two degenerate quark doublets has already been considered in [19]. In next-to-leading order our results show differences to [19], see below.

2 Structure of the theory

2.1 Field definitions

Chiral perturbation theory is formulated in terms of the pseudo-Goldstone fields for spontaneously broken $SU(N_f)_L \otimes SU(N_f)_R$ chiral symmetry. For $N_f = 4$, the 15 canonical fields $\phi_a(x)$, $a = 1 \dots 15$, corresponding to the generators λ_a of $SU(4)$ (see App. A), are assembled in the matrix valued field $\lambda_a \phi_a(x)$. Referring to the usual particle multiplets, the canonical fields are also denoted $(\phi_a) = (\vec{\pi}, \vec{K}, \eta_8, \vec{D}, \vec{D}_s, \eta_{15})$. The fields corresponding to the particle eigenstates are given by the linear combinations

$$\begin{aligned} \pi^\pm &= \frac{1}{\sqrt{2}}(\pi_1 \mp i\pi_2), & \pi^0 &= \pi_3, \\ K^\pm &= \frac{1}{\sqrt{2}}(K_4 \mp iK_5), & K^0, \bar{K}^0 &= \frac{1}{\sqrt{2}}(K_6 \mp iK_7), \\ D^\pm &= \frac{1}{\sqrt{2}}(D_{11} \pm iD_{12}), & D^0, \bar{D}^0 &= \frac{1}{\sqrt{2}}(D_9 \pm iD_{10}), & D_s^\pm &= \frac{1}{\sqrt{2}}(D_{s,13} \pm iD_{s,14}). \end{aligned} \quad (1)$$

The field configuration matrix then reads

$$\frac{1}{\sqrt{2}}\lambda_a\phi_a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 & \pi^+ & K^+ & \bar{D}^0 \\ \pi^- & \frac{-1}{\sqrt{2}}\pi^0 & K^0 & D^- \\ K^- & \bar{K}^0 & 0 & D_s^- \\ D^0 & D^+ & D_s^+ & 0 \end{pmatrix} + \frac{1}{\sqrt{2}}[\lambda_8\eta_8 + \lambda_{15}\eta_{15}]. \quad (2)$$

It enters chiral perturbation theory in terms of the exponential parameterisation $U(x) = \exp\left[\frac{i}{F_0}\lambda_a\phi_a(x)\right]$, where F_0 is a low-energy constant. The effective Lagrangian for $U(x)$ contains chiral symmetry breaking quark mass terms and lattice terms. To leading order (LO) it is given by

$$\mathcal{L}_{\text{LO}} = \frac{F_0^2}{4} \text{Tr}(\partial_\mu U \partial_\mu U^\dagger) - \frac{F_0^2}{4} \text{Tr}(\chi U^\dagger + U \chi^\dagger) - \frac{F_0^2}{4} \text{Tr}(\rho U^\dagger + U \rho^\dagger). \quad (3)$$

Here the quark mass term is

$$\chi = 2B_0 M \quad (4)$$

with the quark mass matrix M , and the lattice artifacts are represented by

$$\rho = 2W_0 a \mathbb{1} , \quad (5)$$

where a is the lattice spacing and B_0 and W_0 are additional low-energy parameters. In next-to-leading order (NLO) the contributions to the effective Lagrangian contain the Gasser-Leutwyler coefficients L_i , as well as new coefficients W_i, W'_i , parameterising the lattice artifacts. We refer the reader to [17, 18] for definitions and details about the NLO Lagrangian.

2.2 Mass terms

In the lattice regularisation, the untwisted physical quark mass m' is proportional to the difference $m_0 - m_{0c}$ of the bare and the critical mass; the prime indicates that the quark mass shift already includes an $\mathcal{O}(a)$ correction aW_0/B_0 , see [17]. In the (u, d) -sector the twisted mass μ is conveniently introduced via the term $i\mu\tau_3$. In chiral perturbation theory we use the variables

$$\chi' = 2B_0(m' + i\mu\tau_3) = 2B_0m_q e^{i\omega_0\tau_3}, \quad m_q = \sqrt{m'^2 + \mu^2}, \quad (6)$$

and

$$\chi'_0 = 2B_0m', \quad \chi_3 = 2B_0\mu, \quad \rho = 2W_0a. \quad (7)$$

In leading order (LO) the pion mass is then given by

$$m_\pi^2 = |\chi'| = \sqrt{\chi_0'^2 + \chi_3^2} = 2B_0m_q. \quad (8)$$

A second twist can be introduced in the (c, s) -doublet in the same way. Considering the definition of the fields in Eq. (2), it should be noted that the twist in the (c, s) -doublet has the opposite sign, because s and c are exchanged so that they are ordered by increasing quark masses in the mass matrix. Following the general argument given in [3, 4], $\mathcal{O}(a)$ -improvement will be realised at twofold maximum twist, i.e. when both twist angles equal $\pi/2$. Maintaining positivity of the fermion determinant requires to implement the twist in the heavy quark sector orthogonal to the mass splitting between the heavy quarks [4, 20]. Because it is more natural to have a diagonal mass matrix, as e.g. in [10], the twist is implemented non-diagonally with τ_1 .

Referring to the definitions above we define for the light and the heavy sector

$$\chi'_{0,l} = 2B_0m'_l, \quad \chi_3 = 2B_0\mu_l, \quad (9)$$

$$\chi'_{0,h} = 2B_0m'_h, \quad \chi_1 = 2B_0\mu_h, \quad \chi_\delta = B_0(m_c - m_s), \quad (10)$$

where m'_h is the average heavy quark mass. The symmetry breaking term χ' can be

written separately for the sectors. With the corresponding Pauli matrices τ_i this reads

$$\chi' = \begin{cases} \chi'_{0,l} + i\chi_3\tau_3 & \text{in } (u, d), \\ \chi'_{0,h} + i\chi_1\tau_1 + \chi_\delta\tau_3 & \text{in } (c, s). \end{cases} \quad (11)$$

The LO twist angles are $\omega_{0,l}$ and $\omega_{0,h}$, respectively; for brevity we define $c_{0,l} = \cos(\omega_{0,l})$, $c_{0,h} = \cos(\omega_{0,h})$ and $s_{0,l} = \sin(\omega_{0,l})$, $s_{0,h} = \sin(\omega_{0,h})$ as in [19].

In the following, perturbative quantities denoted by lower case letters (e.g. m_π^2) are LO and those with capitals (e.g. M_π^2) are NLO.

3 Meson masses in leading order

In the framework defined before, we have calculated meson masses and decay constants in NLO chiral perturbation theory, including lattice artifacts up to $\mathcal{O}(a^2)$. In this article we shall mainly concentrate on meson masses, which are of primary interest. Results on the decay constants can be found in [21].

3.1 Meson masses

For the meson masses in LO we find the extended Gell-Mann-Okubo mass formulae

$$4m_K^2 = 3m_{\eta_8}^2 + m_\pi^2, \quad (12)$$

$$m_K^2 + 9m_D^2 = 6m_{\eta_{15}}^2 + 4m_\pi^2, \quad (13)$$

$$m_K^2 + m_{D_s}^2 = m_{D_s}^2 + m_\pi^2, \quad (14)$$

where the canonical states η_8 and η_{15} are not mass eigenstates of the theory. For the flavour charged mesons we find in terms of the underlying parameters

$$\begin{aligned} m_\pi^2 &= \sqrt{\chi_{0,l}'^2 + \chi_3^2}, \\ m_{D,K}^2 &= \frac{1}{2} \left(\sqrt{\chi_{0,l}'^2 + \chi_3^2} + \sqrt{\chi_{0,h}'^2 + \chi_1^2} \pm \chi_\delta \right), \\ m_{D_s}^2 &= \sqrt{\chi_{0,h}'^2 + \chi_1^2}. \end{aligned} \quad (15)$$

In Monte Carlo calculations, the renormalised physical PCAC quark masses and the bare untwisted quark masses can be obtained from the physical and untwisted lattice currents, respectively, see [10]. With their appropriate renormalisation constants on the lattice

these full non-perturbative masses are related to each other by [10]

$$m_l^{\text{PCAC}} = Z_P^{-1} \sqrt{(Z_A m_{\chi,l}^{\text{PCAC}})^2 + \mu_l^2}, \quad (16)$$

$$\begin{aligned} m_{c,s}^{\text{PCAC}} &= Z_P^{-1} \sqrt{(Z_A m_{\chi,h}^{\text{PCAC}})^2 + \mu_\sigma^2} \pm Z_S^{-1} \mu_\delta, \\ &= m_h^{\text{PCAC}} \pm Z_S^{-1} \mu_\delta. \end{aligned} \quad (17)$$

In terms of the PCAC quark masses the perturbative Eq. (15) reads

$$\begin{aligned} m_\pi^2 &= 2B_0 m_l^{\text{PCAC}}, \\ m_{D,K}^2 &= B_0 m_l^{\text{PCAC}} + B_0 m_{c,s}^{\text{PCAC}}, \\ m_{D_s}^2 &= 2B_0 m_h^{\text{PCAC}}. \end{aligned} \quad (18)$$

3.2 Mixing of flavour neutral mesons

In general there is mixing between the four flavour-neutral mesons π_3 , η_8 and η_{15} from the 15-plet and the SU(4)-singlet η_1 , resulting in the mass eigenstates π^0 , η , η_c and η' . Due to isospin symmetry in our case, π_3 is not mixing with the η 's. In a region with “light” s and c quarks, where chiral perturbation theory holds, the η' can be integrated out and is not included in ordinary chiral perturbation theory.

We define the states η , η_c in the mass diagonal basis through

$$\begin{pmatrix} \eta \\ \eta_c \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_{15} \end{pmatrix}. \quad (19)$$

The mixing angle θ is in LO given by

$$\tan 2\theta = \sqrt{8} \left(1 + \frac{9}{2} \frac{m_D^2 - m_K^2}{m_K^2 - m_\pi^2} \right)^{-1}. \quad (20)$$

The masses of the states in the new basis are then given by

$$\begin{aligned} m_\eta^2 &= m_{\eta_8}^2 - \Delta_{s,c}, \\ m_{\eta_c}^2 &= m_{\eta_{15}}^2 + \Delta_{s,c}, \end{aligned} \quad (21)$$

where $m_{\eta_8}^2$ and $m_{\eta_{15}}^2$ are given by the Gell-Mann-Okubo mass formulae (12), (13), and

$$\Delta_{s,c} = (m_{\eta_{15}}^2 - m_{\eta_8}^2) \frac{\sin^2 \theta}{\cos 2\theta} = \frac{\sqrt{2}}{3} (m_K^2 - m_\pi^2) \tan \theta, \quad (22)$$

which is the analogon to the pion mass splitting in completely non-degenerate 3-flavour QCD [22].

4 Results in next-to-leading order

4.1 Vacuum

In chiral perturbation theory for twisted mass lattice QCD, the vacuum configuration is not given by vanishing fields ϕ_a , corresponding to $U(x) = \mathbb{1}$; instead there is a non-trivial vacuum configuration U_0 , which is obtained by minimising the effective action. The physical fields represent the deviation of $U(x)$ from the vacuum U_0 . It is advantageous to parameterise the physical fields through $U(x) = \sqrt{U_0} U_{\text{phys}}(x) \sqrt{U_0}$. In order to calculate masses and decay constants it is necessary to determine the vacuum U_0 . In our case one has

$$U_0 = \exp \left[\frac{i}{F_0} (\hat{\omega}_l \lambda_3 + \hat{\omega}_h \lambda_{13}) \right]. \quad (23)$$

In NLO we explicitly find

$$\hat{\omega}_l = \omega_{0,l} - \frac{8}{F_0^2} \frac{\rho s_{0,l}}{m_\pi^2} \left\{ m_\pi^2 2W - m_{D_s}^2 [4L_6 - 2W_6] + \rho c_{0,l} 4W' + \rho c_{0,h} 4[L_6 - W_6 + W_6'] \right\}, \quad (24)$$

$$\hat{\omega}_h = \omega_{0,h} - \frac{8}{F_0^2} \frac{\rho s_{0,h}}{m_{D_s}^2} \left\{ m_{D_s}^2 2W - m_\pi^2 [4L_6 - 2W_6] + \rho c_{0,h} 4W' + \rho c_{0,l} 4[L_6 - W_6 + W_6'] \right\}, \quad (25)$$

where $W = W_6 + W_8/2 - 2L_6 - L_8$ and $W' = W_6' + W_8'/2 - W_6 - W_8/2 + L_6 + L_8/2$ (see [17]).

4.2 Pion masses

Calculating the tree-level and one-loop contributions to NLO, we find for the charged pions

$$\begin{aligned} M_\pi^2 = m_\pi^2 &+ \frac{8}{F_0^2} \left\{ m_\pi^4 [2(L_8 + 2L_6) - (L_5 + 2L_4)] + 2m_\pi^2 m_{D_s}^2 [2L_6 - L_4] \right. \\ &- \rho m_\pi^2 c_{0,l} [4(L_8 + 2L_6) - (L_5 + 2L_4) - 2(W_8 + 2W_6) + (W_5 + 2W_4)] \\ &+ \rho m_\pi^2 c_{0,h} 2[L_4 - W_4] \\ &- \rho (m_{D_s}^2 c_{0,l} + m_\pi^2 c_{0,h}) 2[2L_6 - W_6] \\ &+ \rho^2 c_{0,l}^2 2[(L_8 + 2L_6) - (W_8 + 2W_6) + (W_8' + 2W_6')] \\ &\left. + \rho^2 c_{0,l} c_{0,h} 4[L_6 - W_6 + W_6'] \right\} + \text{loop}_\pi. \end{aligned} \quad (26)$$

The loop contribution is

$$\text{loop}_\pi = \frac{m_\pi^2}{N_f (4\pi F_0)^2} \left[2m_\pi^2 \ln \left(\frac{m_\pi^2}{\Lambda^2} \right) - \frac{2 + \xi_1}{3} m_\eta^2 \ln \left(\frac{m_\eta^2}{\Lambda^2} \right) - \frac{1 - \xi_1}{3} m_{\eta_c}^2 \ln \left(\frac{m_{\eta_c}^2}{\Lambda^2} \right) \right] \quad (27)$$

with $\Lambda = 4\pi F_\pi$ conventionally. It depends on

$$\xi_1 \equiv \xi_1(\theta) = \sqrt{2} \sin 2\theta - \sin^2 \theta \in [0, 1]. \quad (28)$$

The expression for M_π^2 explicitly verifies that $\mathcal{O}(a)$ -improvement of the pion mass requires both twist angles to be at maximal twist [3, 4]. In particular, a non-maximal twist angle in the heavy sector would contaminate the pions with $\mathcal{O}(a)$ -terms proportional to $\rho m_\pi^2 c_{0,h}$.

A non-vanishing twist breaks isospin symmetry and leads to a mass splitting between charged and neutral pions. For the mass splitting we find

$$M_\pi^2 - M_{\pi_3}^2 = \frac{16\rho^2 s_{0,l}^2}{F_0^2} [(L_8 + 2L_6) - (W_8 + 2W_6) + (W'_8 + 2W'_6)]. \quad (29)$$

4.3 Kaon masses

The kaons are degenerate at NLO. For their masses we obtain

$$\begin{aligned} M_K^2 = m_K^2 + \frac{4}{F_0^2} \Big\{ & 2m_K^4 [2(L_8 + 4L_6) - (L_5 + 4L_4)] + 4m_K^2 (m_D^2 - m_K^2) [2L_6 - L_4] \\ & - \rho m_K^2 (c_{0,l} + c_{0,h}) [4(L_8 + 4L_6) - (L_5 + 4L_4) - 2(W_8 + 4W_6) + (W_5 + 4W_4)] \\ & - \rho (m_D^2 - m_K^2) (c_{0,l} + c_{0,h}) 2[2L_6 - W_6] \\ & + \rho^2 (c_{0,l} + c_{0,h})^2 [(L_8 + 4L_6) - (W_8 + 4W_6) + (W'_8 + 4W'_6)] \\ & - \rho^2 (s_{0,l}^2 + s_{0,h}^2) [L_8 - W_8 + W'_8] \Big\} + \text{loop}_K \end{aligned} \quad (30)$$

with

$$\text{loop}_K = \frac{m_K^2}{N_f(4\pi F_0)^2} \left[\frac{4 - \xi_2}{3} m_\eta^2 \ln \left(\frac{m_\eta^2}{\Lambda^2} \right) - \frac{1 - \xi_2}{3} m_{\eta_c}^2 \ln \left(\frac{m_{\eta_c}^2}{\Lambda^2} \right) \right] \quad (31)$$

and

$$\xi_2 \equiv \xi_2(\theta) = 5 \sin^2 \theta - \sqrt{2} \sin \theta \cos \theta \in \left[\frac{5-3\sqrt{3}}{2}, 1 \right]. \quad (32)$$

Again, $\mathcal{O}(a)$ improvement is obtained when both light and heavy twists are maximal. In contrast to the case of pions, however, the kaon masses contain $\mathcal{O}(a^2)$ terms at maximal twist.

4.4 Heavy-light case

The heavy-light case is the situation where the both sectors are mass degenerate, i.e. $m_l = m_u = m_d$ and $m_h = m_s = m_c$. In this special case the mixing angle is analytically given by

$$\tan 2\theta = \sqrt{8} \quad (33)$$

This mixing angle corresponds to including the physical states η and η_c in the field matrix by replacing the generators λ_8, λ_{15} by $\lambda_8^*, \lambda_{15}^*$ (see App. A).

In LO the mass spectrum given in Eqs. (15), (18), (21) simplifies to

$$\begin{aligned} m_\pi^2 &= 2B_0 m_l^{\text{PCAC}}, \\ m_K^2 &= m_D^2 = m_\eta^2 = B_0 m_h^{\text{PCAC}} + B_0 m_l^{\text{PCAC}}, \\ m_{D_s}^2 &= m_{\eta_c}^2 = 2B_0 m_h^{\text{PCAC}}. \end{aligned} \quad (34)$$

The loop contributions simplify too, because of

$$\xi_1(\theta) = \xi_2(\theta) = 1. \quad (35)$$

In NLO there are mixing effects between the K - and D -mesons, when $m_s = m_c$. In the generic case $m_s \neq m_c$ these mixings contribute only at NNLO, related to the non-analytic behaviour in $m_s - m_c$. In the case $m_s = m_c$, however, the mixing already shows up in NLO. In particular it leads to a flavour breaking in both sectors according to

$$M_{K^\pm}^2 - M_{K^0}^2 = M_{D^\pm}^2 - M_{D^0}^2 = \frac{16\rho^2 s_{0,l} s_{0,h}}{F_0^2} [L_8 - W_8 + W_8'], \quad (36)$$

and the above formula (30) for M_K^2 gives the average of the kaon masses. This breaking is a pure lattice artifact.

At tree level the resulting formulae for the kaon masses coincide exactly with those given in [19]. In NLO, however, the loop contributions to the kaon masses

$$\frac{m_K^2}{N_f(4\pi F_0)^2} \left[m_\eta^2 \ln \left(\frac{m_\eta^2}{\Lambda^2} \right) \right], \quad (37)$$

which we obtain, are different from those given in [19]:

$$\frac{m_K^2}{N_f(4\pi F_0)^2} \left[\frac{4}{3} m_{\eta_8}^2 \ln \left(\frac{m_{\eta_8}^2}{\Lambda^2} \right) - \frac{1}{3} m_{\eta_{15}}^2 \ln \left(\frac{m_{\eta_{15}}^2}{\Lambda^2} \right) \right]. \quad (38)$$

The origin of this discrepancy is the neglect of the mixing of the η -mesons in [19]. The results are only valid if the mixing disappears, which only applies if all quark masses are degenerate. Also, as a consequence, the loop-divergences in [19] do not cancel the tree-level ones if $m_l \neq m_h$.

A related discrepancy can be found for the kaon decay constant, where our result for the loop contribution is

$$\frac{N_f}{2(4\pi F_0)^2} \left[\frac{3}{16} m_\pi^2 \ln \left(\frac{m_\pi^2}{\Lambda^2} \right) + \frac{5}{8} m_K^2 \ln \left(\frac{m_K^2}{\Lambda^2} \right) + \frac{3}{16} m_{D_s}^2 \ln \left(\frac{m_{D_s}^2}{\Lambda^2} \right) \right] \quad (39)$$

in contrast to [19]:

$$\frac{N_f}{2(4\pi F_0)^2} \left[\frac{3}{16} m_\pi^2 \ln \left(\frac{m_\pi^2}{\Lambda^2} \right) + \frac{1}{2} m_K^2 \ln \left(\frac{m_K^2}{\Lambda^2} \right) + \frac{3}{16} m_{\eta_8}^2 \ln \left(\frac{m_{\eta_8}^2}{\Lambda^2} \right) + \frac{1}{8} m_{D_s}^2 \ln \left(\frac{m_{D_s}^2}{\Lambda^2} \right) \right]. \quad (40)$$

5 Conclusion

We have obtained the masses of pseudoscalar mesons in the framework of chiral perturbation theory for twisted mass lattice QCD with a degenerate doublet of u and d quarks and a non-degenerate doublet of s and c quarks in next-to-leading order, including lattice effects up to $\mathcal{O}(a^2)$. The results display the dependence of the masses on the two twist angles for the light and heavy sectors. For maximal twist in both sectors, Symanzik improvement to $\mathcal{O}(a)$, as proven in [3, 4], is explicitly exhibited. The mixing of flavour-neutral mesons is also discussed. For the case of degenerate s and c quarks, proper account of mixing corrects results in the literature.

A Generators of SU(N)

The generators of SU(N) in the fundamental representation are the hermitian $N \times N$ matrices with zero trace. A useful representation is as follows [23]. Let e_{ij} be the matrix with the only non-vanishing entry being 1 in row/column ij :

$$(e_{ij})_{kl} = \delta_{ik} \delta_{jl}. \quad (41)$$

The off-diagonal generators of SU(N) can be written in terms of $N \times N$ generalisations of the Pauli matrices σ_1 and σ_2 , defined by

$$\begin{aligned} \sigma_{1;ij} &= e_{ij} + e_{ji}, & (i < j), \\ \sigma_{2;ij} &= -ie_{ij} + ie_{ji}, & (i < j). \end{aligned} \quad (42)$$

The diagonal traceless matrices, generalising σ_3 , are defined by

$$\sigma_{3;i} = \sqrt{\frac{2}{i(i-1)}} \operatorname{diag}(\underbrace{1, \dots, 1}_{i-1}, 1-i, \underbrace{0, \dots, 0}_{N-i}), \quad (i > 1). \quad (43)$$

The extended Gell-Mann matrices λ_a are then given by

$$\begin{aligned} \sigma_{1;ij} &= \lambda_a, & \text{for } a = (j-1)^2 + 2i - 2, & (i < j), \\ \sigma_{2;ij} &= \lambda_a, & \text{for } a = (j-1)^2 + 2i - 1, & (i < j), \\ \sigma_{3;i} &= \lambda_a, & \text{for } a = i^2 - 1, & (i > 1), \end{aligned} \quad (44)$$

where $a = 1, \dots, N^2 - 1$, and the generators of $SU(N)$ are equal to $\lambda_a/2$. They are orthonormal in the sense of

$$\text{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}. \quad (45)$$

For $SU(4)$ the diagonal Gell-Mann matrices, corresponding to the flavour neutral mesons, are explicitly given by

$$\lambda_3 = \text{diag}(1, -1, 0, 0), \quad \sqrt{3}\lambda_8 = \text{diag}(1, 1, -2, 0), \quad \sqrt{6}\lambda_{15} = \text{diag}(1, 1, 1, -3). \quad (46)$$

In the heavy-light case the latter two can be replaced for simplicity with the linear combinations

$$\sqrt{2}\lambda_8^* = \text{diag}(1, 1, -1, -1), \quad \lambda_{15}^* = \text{diag}(0, 0, 1, -1). \quad (47)$$

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